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Second variation of the Helfrich–Canham Hamiltonian and reparametrization invariance

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Abstract

A covariant approach towards a theory of deformations is developed to examine both the first and second variation of the Helfrich–Canham Hamiltonian—quadratic in extrinsic curvature—which describes fluid vesicles at mesoscopic scales. Deformations are decomposed into tangential and normal components. At first order, tangential deformations may always be identified with a reparametrization; at second order, they differ. The relationship between tangential deformations and reparametrizations, as well as the coupling between tangential and normal deformations, is examined at this order for both the metric and the extrinsic curvature tensors. Expressions for the expansion to second order in deformations of geometrical invariants constructed with these tensors are obtained; in particular, the expansion of the Hamiltonian to this order about an equilibrium is considered. Our approach applies as well to any geometrical model for membranes.

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1. Introduction

In water, lipid molecules assemble spontaneously into vesicles which are described remarkably well at mesoscopic scales by a purely geometrical Hamiltonian [1–3]. On such scales, there is a difference of several orders of magnitude between the thickness of the lipid bilayer and the diameter of the vesicle; it is therefore sensible to describe the vesicle itself as a two-dimensional surface, the relevant coarse grained degrees of freedom are purely geometrical and describe the shape of this surface. Furthermore, this membrane acts like a two-dimensional fluid: there is no cost in energy associated with tangential displacements of

the lipid constituents which preserve the area, and thus its shear modulus vanishes. In this respect, the membrane differs completely in its behaviour from a familiar elastic solid. As a two-dimensional fluid the membrane is then described by an effective energy that does not penalize tangential displacements. Infinitesimally, tangential displacements can be identified with a reparametrization of the surface. The appropriate Hamiltonian must therefore be a geometrical invariant under reparametrizations.

The Helfrich–Canham Hamiltonian quadratic in the mean extrinsic curvature describes the penalty associated with the bending of the vesicle [4–6]. The microscopic physics of the lipid molecules is encoded in the rigidity modulus characterizing the stiffness of the membrane on the particular mesoscopic scale being considered.

There are global constraints on the shape of the vesicle: the area is fixed, and on time scales relevant in experiments, the enclosed volume is also fixed. At first order, the particular composition of the lipid bilayer, and in particular the asymmetry between the layers, is characterized by a constraint or a penalty on the total mean extrinsic curvature of the surface, a quantity which captures the area difference between the two layers. For the sake of simplicity we will restrict our attention to a minimal geometric model for fluid vesicles which is known as the strict bilayer couple model [7]. A more realistic geometric model, the area-difference model, takes the bilayer composition into account more precisely, in particular, the difference in stretching of the individual layers [8–10]. The two models are related by a Legendre transformation; the formal questions we address apply to both.

Our aim in this paper is to provide an approach towards a *covariant* theory of deformations of a membrane described by the Helfrich–Canham Hamiltonian, although our considerations will not depend on the details of this model; they apply equally well to any reparametrization invariant geometrical theory of membranes. The basic variables are the shape functions describing the surface. We examine how the geometry of this surface changes under a deformation. The approach we will adopt complements the one presented in [11] where the deformation was decomposed into its tangential and normal components with a focus on the latter; it is similar, at least in spirit, to the approach taken by Cai and Lubensky in their description of membrane dynamics [12, 13].

At first order in the deformation, the change in the Hamiltonian vanishes when the vesicle is in equilibrium. At this order, the tangential deformation of the Hamiltonian appears only in a boundary term, so that for the purposes of examining the equilibria of closed vesicles, it can always be neglected. This is consistent with our understanding that an infinitesimal tangential deformation is a reparametrization of the surface. In contrast to what one might expect, however, this identification breaks down at higher orders. Finite tangential deformations are *not* simple exponentials of infinitesimal reparametrizations. Nonetheless, as was shown in [11], if one is interested only in fluctuations about equilibrium (so that the Euler–Lagrange equation is satisfied), tangential deformations remain irrelevant at second order. This is because, at this order, the tangential contribution to the deformation of any given term appearing in the Hamiltonian is proportional to the Euler–Lagrange derivative of that term. In equilibrium the sum of these terms vanishes: the second variation of the total Hamiltonian about equilibrium is thus always quadratic in the normal deformation. This is the principal justification for the cavalier approach adopted in [11] where tangential deformations are discarded from the outset; considering the effort one must expend to keep track of tangential contributions, the fact that no error is incurred represents a stroke of good luck. What it fails to do, however, even at second order is to provide a correct expansion of individual geometrical tensors, such as the metric and the extrinsic curvatures: tangential deformations not only contribute but also couple non-trivially to normal deformations and there is no justification to drop them.

In this paper, we will examine the coupling between tangential and normal components at second and higher orders. We will also attempt to quantify the extent to which tangential deformations differ from reparametrizations. These issues do not appear to have been addressed before. Besides their value in point of principle, there is also a practical value in understanding them: there are occasions when it is necessary to look beyond second order. To identify the stable deformations of a spherical equilibrium shape one needs to expand the Hamiltonian out to fourth order [14, 15] in order to resolve a degeneracy occurring at second order. Helfrich and Ou-Yang did not attempt the full calculation, focusing instead on a single mode. Whereas at second order they can be ignored, tangential deformations will contribute at higher orders to perturbations about an equilibrium, and thus they must be confronted. It is perhaps not too surprising that, to date, a renormalization group analysis of the fluid membrane model has not been attempted at two loops. The number of terms involved, even in a straightforward Monge representation of deformations, is sufficient to discourage the faint-hearted. To attempt such an exercise, with some hope of successfully completing it, it is important to identify the underlying patterns in the expansion of the Hamiltonian.

This paper is organized as follows. Section 2 introduces the geometry of two-dimensional surfaces and the Helfrich–Canham Hamiltonian for lipid membranes. Sections 3 and 4 consider how the intrinsic and extrinsic geometries of the surface change under an infinitesimal covariant deformation up to second order, respectively. In section 5, the deformation is decomposed into its normal and tangential components and we examine the relationship between tangential deformations and reparametrizations both at first and second order. In section 6, we derive the first-order variation of the Helfrich–Canham Hamiltonian and we identify its Euler–Lagrange derivatives, obtaining the shape equation which determines the equilibrium configurations. In section 7, we derive general expressions for the stresses and torques associated with the Hamiltonian, providing an alternative derivation of the results of [16]. The second-order variation of the Hamiltonian is derived in section 8. We conclude in section 9 with some final remarks.

2. Geometric model

We model a lipid vesicle as a two-dimensional surface Σ embedded in three-dimensional space. This surface is specified locally in parametric form by three shape functions $\mathbf{X} = (X^1, X^2, X^3)$, $\mathbf{x} = \mathbf{X}(\xi^a)$, where the coordinates $\mathbf{x} = x^\mu = (x^1, x^2, x^3)$ describe a point in space, $\xi^a = (\xi^1, \xi^2)$ are arbitrary coordinates on the surface.

First, we recall briefly some basic facts about the geometry of surfaces. For a thorough introduction to this subject see e.g. [17, 18]. The two tangent vectors to Σ are $\mathbf{e}_a = \partial_a \mathbf{X}$, with $\partial_a = \partial/\partial\xi^a$. The metric induced on Σ by the embedding is defined by $g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$. Latin indices are lowered and raised with g_{ab} and its inverse g^{ab}, respectively. The induced metric defines the infinitesimal area element with $\mathrm{d}A = \sqrt{g}\,\mathrm{d}^2\xi$, where g denotes the determinant of g_{ab}. The unit normal to the surface Σ, $\mathbf{n}$, is defined implicitly by $\mathbf{n} \cdot \mathbf{e}_a = 0$, and $\mathbf{n} \cdot \mathbf{n} = 1$. We note that the basis vectors $\{\mathbf{e}_a, \mathbf{n}\}$ are complete: given any two vectors $\mathbf{U}$ and $\mathbf{V}$,

$$\mathbf{U} \cdot \mathbf{V} = (\mathbf{U} \cdot \mathbf{n})(\mathbf{V} \cdot \mathbf{n}) + (\mathbf{U} \cdot \mathbf{e}_a)(\mathbf{V} \cdot \mathbf{e}^a). \tag{1}$$

The classical Gauss–Weingarten equations describe the expansion of the surface gradients of the basis $\{\mathbf{e}_a, \mathbf{n}\}$ adapted to the surface Σ in terms of the basis:

$$\partial_a \mathbf{e}_b = \Gamma^c_{ab} \mathbf{e}_c - K_{ab}\mathbf{n} \tag{2}$$

$$\partial_a \mathbf{n} = K_{ab} g^{bc} \mathbf{e}_c. \tag{3}$$

Here Γ^c_{ab} denotes the Christoffel symbols of the Σ covariant derivative compatible with g_{ab}, such that for an arbitrary surface vector V^b we have $\nabla_a V^b = \partial_a V^b + \Gamma^b_{ac} V^c$. By compatible we mean $\nabla_a g_{bc} = 0$. The Christoffel symbols Γ^c_{ab} are given in terms of the induced metric by

$$\Gamma^a{}_{bc} = \mathbf{e}^a \cdot \partial_b \mathbf{e}_c = \tfrac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}). \tag{4}$$

Geometrically, the Γ^c_{ab} are purely intrinsic: they depend only on the induced metric g_{ab}. In a geometrical covariant description of the surface, the Christoffel symbols appear only through the covariant derivative. The intrinsic Riemann curvature tensor of Σ quantifies the degree of failure of the covariant derivative ∇_a to commute,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = \mathcal{R}^c{}_{dab} V^d. \tag{5}$$

In terms of the Christoffel symbols, the Riemann tensor takes the form

$$\mathcal{R}^a{}_{bcd} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^a_{ce} \Gamma^e{}_{db} - \Gamma^a_{de} \Gamma^e_{cb}. \tag{6}$$

The contraction of the Riemann tensor gives the Ricci tensor $\mathcal{R}_{ab} = \mathcal{R}^c{}_{acb}$, and the scalar curvature is given by contraction with the contravariant metric $\mathcal{R} = g^{ab} \mathcal{R}_{ab}$. For a two-dimensional surface, the Riemann tensor is completely determined by the scalar curvature

$$\mathcal{R}_{abcd} = (\mathcal{R}/2)(g_{ac} g_{bd} - g_{ad} g_{bc}), \tag{7}$$

which implies, in particular, $\mathcal{R}_{ab} = \frac{1}{2} \mathcal{R} g_{ab}$. The scalar curvature of a two-dimensional surface is twice the Gaussian curvature G, i.e. $\mathcal{R} = 2\mathrm{G}$.

The extrinsic curvature tensor of the surface Σ is

$$K_{ab} = -\mathbf{n} \cdot \partial_a \mathbf{e}_b = K_{ba}. \tag{8}$$

As a real symmetric two by two matrix it can always be diagonalized. In particular, the eigenvalues c_1, c_2 of the matrix $K_a{}^b = g^{bc} K_{ca}$ are the principal curvatures of the surface. The trace with the contravariant metric $K = g^{ab} K_{ab}$ is the mean curvature of the surface. With respect to the principal curvatures $K = c_1 + c_2$. In the literature, often the mean curvature is denoted by $H = \frac{1}{2} K$. The Gaussian curvature is given in terms of the principal curvatures by their product, $G = c_1 c_2$.

The intrinsic and extrinsic geometries of Σ are related by the Gauss–Codazzi–Mainardi equations, which arise as integrability conditions for the Gauss–Weingarten equations (2), (3),

$$\mathcal{R}_{abcd} - K_{ac} K_{bd} + K_{ad} K_{bc} = 0 \tag{9}$$

$$\nabla_a K_{bc} - \nabla_b K_{ac} = 0. \tag{10}$$

We will use extensively their contractions with the contravariant metric g^{ab}:

$$\mathcal{R}_{ab} - K K_{ab} + K_{ac} K_b{}^c = 0 \tag{11}$$

$$\mathcal{R} - K^2 + K_{ab} K^{ab} = 0 \tag{12}$$

$$\nabla_b K_a{}^b - \nabla_a K = 0. \tag{13}$$

Note that for a two-dimensional surface the contracted Gauss–Codazzi equation (12) contains the same information as (9).

The fluid state of the lipid vesicle implies that in an effective mesoscopic description shear is negligible, therefore the vesicle Hamiltonian has to be invariant under reparametrizations. Moreover, as recognized long ago the important mode of deformation is out of the surface, corresponding to a bending [4–6]. The bending energy is quadratic in the mean extrinsic curvature

$$F_b = \alpha \int \mathrm{d}A \, K^2 \tag{14}$$

where the constant α denotes the bending rigidity. At the same order, we have also the Gaussian bending energy

$$F_G = \alpha_G \int \mathrm{d}A\, \mathcal{R} \tag{15}$$

with α_G being the Gaussian bending rigidity. However, if the surface has no boundary, by the Gauss–Bonnet theorem, the Gaussian bending energy is a topological invariant (see e.g. [18]):

$$F_G = 8\pi\alpha_G(1 - \mathrm{g}) \tag{16}$$

where g is the genus of the surface. As such it does not contribute to the determination of the equilibrium configurations of the membrane. Note that at the same order we have also the geometrical invariant $\int \mathrm{d}A\, K^{ab}K_{ab}$. However, it is not independent since it is related to the bending and Gaussian bending energies via the Gauss–Codazzi equation (12).

The lipid vesicle is subject to various geometric constraints. The low solubility of the lipid molecules implies that its area A is constant. The low permeability of the membrane implies that the enclosed volume V is constant. We write the enclosed volume V as a surface integral with

$$V = \frac{1}{3}\int \mathrm{d}A\, \mathbf{n}\cdot\mathbf{X}. \tag{17}$$

The bilayer architecture of the lipid membrane is captured, in a first approximation, by a constraint on the area difference between the layers. This is expressed as a constant total mean curvature (see [7])

$$M = \int \mathrm{d}A\, K \tag{18}$$

since, as we shall see below, the difference in area is proportional to the mean extrinsic curvature.

Therefore we are led to consider the Helfrich–Canham Hamiltonian

$$F = F_b + \mu A + \beta M - PV \tag{19}$$

where μ, P, β are the Lagrange multipliers that enforce the constraints of constant area, volume and constant mean extrinsic curvature, or constant area difference, respectively.

It is important to emphasize that a more realistic model of the lipid vesicle which takes the bilayer architecture more accurately into account is the area-difference model [8–10]. A thorough discussion of the extant curvature models for lipid membranes can be found, e.g., in the reviews [3, 19, 20].

3. Deformations of the intrinsic geometry

Let us consider now deformations of the surface Σ. A one-parameter infinitesimal deformation of the shape functions $\mathbf{X}(\xi^a)$ can be described by

$$\widetilde{\mathbf{X}}(\xi^a) = \mathbf{X}(\xi^a) + \epsilon\mathbf{W}(\xi^a). \tag{20}$$

$\mathbf{W}(\xi^a)$ is an arbitrary vector field, and the constant ϵ is an infinitesimal parameter. Such a deformation gives us a new surface $\widetilde{\Sigma}$. We begin by examining how the intrinsic geometry of the two surfaces $\widetilde{\Sigma}$ and Σ are connected up to second order in ϵ. A tilde will be used to denote the geometrical quantities that characterize $\widetilde{\Sigma}$. The content of this section can be found in many monographs on differential geometry. However, it is often presented in an abstract notation quite unfamiliar to the working physicist. For this reason, we offer a self-contained derivation of the relationship between the geometries of Σ and $\widetilde{\Sigma}$.

We will not consider deformations in which $\mathbf{W}$ itself depends explicitly on $\mathbf{X}$, such as rigid rotations of the surface or contexts where it is useful to tie the surface local coordinates ξ^a to the embedding itself. An example of the latter is to parametrize the surface by arclength along some privileged directions. Consistency would require then that the coordinates themselves suffer a deformation. Failure to account for this fact is a source of frequent errors in the literature.

At *fixed* values of the arbitrary surface coordinates ξ^a, the tangent vectors to Σ and $\widetilde{\Sigma}$ are related by

$$\widetilde{\mathbf{e}}_a = \mathbf{e}_a + \mathbf{e}_{a(1)} = \mathbf{e}_a + \epsilon \mathbf{W}_a \tag{21}$$

where we define $\mathbf{W}_a = \partial_a \mathbf{W}$. The change in the tangent vectors is only linear in ϵ, like the shape functions $\mathbf{X}$ themselves. For a constant deformation, $\mathbf{W} = \mathbf{a} = \text{const}$, the tangent vectors coincide. A translation of the surface will not change its geometry.

It follows, using (21), that the induced metric on $\widetilde{\Sigma}$ takes the form

$$\widetilde{g}_{ab} = \widetilde{\mathbf{e}}_a \cdot \widetilde{\mathbf{e}}_b = g_{ab} + 2\epsilon(\mathbf{e}_{(a} \cdot \mathbf{W}_{b)}) + \epsilon^2(\mathbf{W}_a \cdot \mathbf{W}_b) \tag{22}$$

where the parentheses enclosing indices denote symmetrization, i.e. $A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba})$. It should be emphasized that this expression is valid to all orders in ϵ; it terminates at second order.

The area measure on $\widetilde{\Sigma}$ is $\mathrm{d}\widetilde{A} = \sqrt{\widetilde{g}}\,\mathrm{d}^2\xi$, with $\widetilde{g}$ being the determinant of $\widetilde{g}_{ab}$. It is related to the area measure on Σ by the expression

$$\sqrt{\widetilde{g}} = \sqrt{g}\left\{1 + \epsilon(\mathbf{e}_a \cdot \mathbf{W}^a) + \frac{\epsilon^2}{2}[(\mathbf{n} \cdot \mathbf{W}^a)(\mathbf{n} \cdot \mathbf{W}_a) + (\mathbf{e}_a \cdot \mathbf{W}^a)(\mathbf{e}_b \cdot \mathbf{W}^b) - (\mathbf{e}_b \cdot \mathbf{W}^a)(\mathbf{e}_a \cdot \mathbf{W}^b)]\right\} + \mathcal{O}(\epsilon^3). \tag{23}$$

Note that the two terms on the second line have the structure of a determinant for the two-dimensional matrix $(\mathbf{e}_a \cdot \mathbf{W}^b)$. In order to derive this expression, we need the inverse induced metric $\widetilde{g}^{ab}$, defined by $\widetilde{g}_{ac}\widetilde{g}^{bc} = \delta^b_a$, together with (22). For this purpose, we expand $\widetilde{g}^{ab}$ and $\widetilde{g}_{ab}$ in powers of ϵ. Collecting terms linear in ϵ, we have the condition $g^{ab}{}_{(1)}g_{bc} + g^{ab}g_{bc(1)} = 0$, where the number in parentheses refers to the order in ϵ. This gives

$$g^{ab}{}_{(1)} = -2\epsilon(\mathbf{e}^{(a} \cdot \mathbf{W}^{b)}). \tag{24}$$

At second order in ϵ, we have the condition $g^{ab}{}_{(2)}g_{bc} + g^{ab}{}_{(1)}g_{bc(1)} + g^{ab}g_{bc(2)} = 0$, which, in turn, yields

$$g^{ab}{}_{(2)} = \epsilon^2[(\mathbf{e}^a \cdot \mathbf{W}_c)(\mathbf{e}^b \cdot \mathbf{W}^c) + 2(\mathbf{e}^{(a} \cdot \mathbf{W}_c)(\mathbf{W}^{b)} \cdot \mathbf{e}^c) - (\mathbf{n} \cdot \mathbf{W}^a)(\mathbf{n} \cdot \mathbf{W}^b)] \tag{25}$$

where we have used the completeness relation (1) to get $(\mathbf{e}_c \cdot \mathbf{W}^a)(\mathbf{e}^c \cdot \mathbf{W}^b) = (\mathbf{W}^a \cdot \mathbf{W}^b) - (\mathbf{n} \cdot \mathbf{W}^a)(\mathbf{n} \cdot \mathbf{W}^b)$. It follows that for the inverse induced metric on $\widetilde{\Sigma}$ we have

$$\widetilde{g}^{ab} = g^{ab} - 2\epsilon(\mathbf{e}^{(a} \cdot \mathbf{W}^{b)}) + \epsilon^2[(\mathbf{e}^a \cdot \mathbf{W}_c)(\mathbf{e}^b \cdot \mathbf{W}^c) - (\mathbf{n} \cdot \mathbf{W}^a)(\mathbf{n} \cdot \mathbf{W}^b) + 2(\mathbf{e}^{(a} \cdot \mathbf{W}_c)(\mathbf{W}^{b)} \cdot \mathbf{e}^c)] + \mathcal{O}(\epsilon^3). \tag{26}$$

Note that $\widetilde{g}^{ab}$, unlike $\tilde{g}_{ab}$, has corrections at all orders in ϵ.

We now compute $\sqrt{\widetilde{g}}$ via a Taylor expansion in ϵ,

$$\sqrt{\widetilde{g}} = \sqrt{g} + \sqrt{g}_{(1)} + \sqrt{g}_{(2)} + \mathcal{O}(\epsilon^3) = \sqrt{g} + \epsilon\left[\frac{\partial\sqrt{\widetilde{g}}}{\partial\epsilon}\right]_{\epsilon=0} + \frac{\epsilon^2}{2}\left[\frac{\partial^2\sqrt{\widetilde{g}}}{\partial\epsilon^2}\right]_{\epsilon=0} + \mathcal{O}(\epsilon^3). \tag{27}$$

We calculate

$$\frac{\partial\sqrt{\widetilde{g}}}{\partial\epsilon} = \frac{1}{2}\sqrt{\widetilde{g}}\widetilde{g}^{ab}\frac{\partial\widetilde{g}_{ab}}{\partial\epsilon} = \sqrt{\widetilde{g}}[\widetilde{g}^{ab}(\mathbf{e}_a\cdot\mathbf{W}_b) + \epsilon\widetilde{g}^{ab}(\mathbf{W}_a\cdot\mathbf{W}_b)]. \tag{28}$$

Taking the $\epsilon\to 0$ limit, we have

$$\sqrt{g}_{(1)} = \epsilon\sqrt{g}(\mathbf{e}_a\cdot\mathbf{W}^a). \tag{29}$$

At second order, we have

$$\frac{\partial^2\sqrt{\widetilde{g}}}{\partial\epsilon^2} = \left(\frac{\partial\sqrt{\widetilde{g}}}{\partial\epsilon}\right)[\widetilde{g}^{ab}(\mathbf{e}_a\cdot\mathbf{W}_b) + \epsilon(\mathbf{W}_a\cdot\mathbf{W}_b)] + \sqrt{\widetilde{g}}\,\widetilde{g}^{ab}(\mathbf{W}_a\cdot\mathbf{W}_b) + \sqrt{\widetilde{g}}\left(\frac{\partial\widetilde{g}^{ab}}{\partial\epsilon}\right)[(\mathbf{e}_a\cdot\mathbf{W}_b) + \epsilon(\mathbf{W}_a\cdot\mathbf{W}_b)]. \tag{30}$$

Using (26) and (28) in this expression, and taking the $\epsilon\to 0$ limit give the second-order correction in (23).

Finally, we consider the intrinsic scalar curvature $\widetilde{R}$. We restrict our attention to the first-order correction. We use the Palatini identity for the first-order correction of the Christoffel symbols Γ^c_{ab},

$$\Gamma^c_{ab(1)} = \tfrac{1}{2}g^{cd}(\nabla_b g_{ad(1)} + \nabla_a g_{bd(1)} - \nabla_d g_{ab(1)}) \tag{31}$$

to obtain, using (22),

$$\begin{aligned}\Gamma^c_{ab(1)} &= \epsilon(\mathbf{e}^c\cdot\nabla_a\mathbf{W}_b + \mathbf{W}^c\cdot\nabla_a\mathbf{e}_b)\\ &= \epsilon[\mathbf{e}^c\cdot\nabla_a\mathbf{W}_b - K_{ab}(\mathbf{W}^c\cdot\mathbf{n})]\end{aligned} \tag{32}$$

where we have used the Gauss–Weingarten equation (2) in the second line, and the fact that the covariant derivative is torsionless, i.e. $\nabla_a\mathbf{W}_b = \nabla_b\mathbf{W}_a$. We recall that $\Gamma^c_{ab(1)}$, unlike Γ^c_{ab}, transforms as a tensor under surface reparametrizations. Using the definition of the Riemann tensor given by (6), we have that at first order the Riemann tensor is

$$\mathcal{R}^a{}_{bcd(1)} = \nabla_c\Gamma^a_{db(1)} - \nabla_d\Gamma^a_{cb(1)} \tag{33}$$

so that, inserting (32), we have

$$\begin{aligned}\mathcal{R}^a{}_{bcd(1)} = \epsilon\big[&-\mathcal{R}^e{}_{bcd}(\mathbf{e}^a\cdot\mathbf{W}_e) + \mathcal{R}_{cdb}{}^e(\mathbf{e}_e\cdot\mathbf{W}^a) + K_{bc}(\mathbf{n}\cdot\nabla_d\mathbf{W}^a)\\ &- K_c{}^a(\mathbf{n}\cdot\nabla_d\mathbf{W}_b) + K_d{}^a(\mathbf{n}\cdot\nabla_c\mathbf{W}_b) - K_{db}(\mathbf{n}\cdot\nabla_c\mathbf{W}^a)\big]\end{aligned} \tag{34}$$

where we have used the definition of the Riemann tensor and both the Codazzi–Mainardi equations (10) and the Gauss–Codazzi equations (9). For the Ricci tensor this implies

$$\begin{aligned}\mathcal{R}_{bd(1)} &= \nabla_c\Gamma^c_{bd(1)} - \nabla_d\Gamma^c_{cb(1)}\\ &= \epsilon\big[-\big(\mathcal{R}^e{}_{bad} + \mathcal{R}^e{}_{dab}\big)(\mathbf{e}^a\cdot\mathbf{W}_e) + K_{ab}(\mathbf{n}\cdot\nabla_d\mathbf{W}^a)\\ &\quad + K_a{}^a(\mathbf{n}\cdot\nabla_a\mathbf{W}_b) - K_{db}(\mathbf{n}\cdot\nabla_c\mathbf{W}^c) - K(\mathbf{n}\cdot\nabla_b\mathbf{W}_d).\end{aligned} \tag{35}$$

These expressions are valid for a hypersurface of arbitrary dimension. Restricting our attention to two-dimensional surfaces, and exploiting the fact that both the Riemann tensor and the Ricci tensor can be expressed in terms of the scalar curvature $\mathcal{R}$, we have that

$$g^{ab}\mathcal{R}_{ab(1)} = \epsilon[2(K^{ab} - Kg^{ab})(\mathbf{n}\cdot\nabla_a\mathbf{W}_b) - \mathcal{R}(\mathbf{e}_a\cdot\mathbf{W}^a)]. \tag{36}$$

For the scalar curvature we have then

$$\begin{aligned}\mathcal{R}_{(1)} &= g^{ab}{}_{(1)}\mathcal{R}_{ab} + g^{ab}\mathcal{R}_{ab(1)}\\ &= 2\epsilon[(K^{ab} - Kg^{ab})(\mathbf{n}\cdot\nabla_a\mathbf{W}_b) - \mathcal{R}(\mathbf{e}^a\cdot\mathbf{W}_a)].\end{aligned} \tag{37}$$

Note that it depends on two derivatives of the deformation vector $\mathbf{W}$, and that it involves the extrinsic geometry of the surface.

In general, it is always possible to expand any geometrical quantity $\widetilde{f}$ on the deformed surface $\widetilde{\Sigma}$ in terms of $\mathbf{W}$. With $\widetilde{f} = f + f_{(1)} + f_{(2)}$, besides direct computation, we can obtain the second-order term $f_{(2)}$ by 'deforming' the first-order term $f_{(1)}$, that is via the important identity

$$f_{(2)} = \tfrac{1}{2}\widetilde{f}_{(1)}. \tag{38}$$

Here $\widetilde{f}_{(1)}$ is to be understood as the expansion to order ϵ of $f_{(1)}$ as we illustrate below. This alternative approach is particularly useful when we consider global geometric quantities associated with $\widetilde{\Sigma}$ (see sections 6 and 8). Note that, in agreement with the identity (38), we have, for example,

$$\begin{aligned}\sqrt{g}_{(2)} = \frac{\epsilon}{2}\widetilde{\sqrt{g}}_{(1)} &= \frac{\epsilon}{2}\big[\sqrt{g}_{(1)}(\mathbf{e}^a\cdot\mathbf{W}_a) + \sqrt{g}\,g^{ab}_{(1)}(\mathbf{e}_a\cdot\mathbf{W}_b) + \sqrt{g}(\mathbf{W}_a\cdot\mathbf{W}^a)\big]\\ &= \frac{\epsilon^2}{2}\sqrt{g}[(\mathbf{e}^a\cdot\mathbf{W}_a)^2 - (\mathbf{e}^a\cdot\mathbf{W}^b)(\mathbf{e}_a\cdot\mathbf{W}_b) - (\mathbf{e}^b\cdot\mathbf{W}^a)(\mathbf{e}_a\cdot\mathbf{W}_b)\\ &\quad + (\mathbf{W}_a\cdot\mathbf{W}^a)]\end{aligned} \tag{39}$$

which, using the completeness relation (1) in the last term, reproduces the second-order contribution to (23). The identity (38) can be proved using variational techniques. It does not appear to be available in the literature. Of course, this could be just a shortcoming of our search.

4. Deformation of the extrinsic geometry

Let us turn now to the extrinsic geometry of the deformed surface $\widetilde{\Sigma}$. For its unit normal $\widetilde{\mathbf{n}}$, we use the defining relations $\widetilde{\mathbf{n}}\cdot\widetilde{\mathbf{e}}_a = 0, \widetilde{\mathbf{n}}\cdot\widetilde{\mathbf{n}} = 1$, together with (21). We expand $\widetilde{\mathbf{n}}$ and we obtain the relations

$$\begin{aligned}&\epsilon(\mathbf{n}\cdot\mathbf{W}_a) + (\mathbf{n}_{(1)}\cdot\mathbf{e}_a) = 0 \qquad && \epsilon(\mathbf{n}_{(1)}\cdot\mathbf{W}_a) + (\mathbf{n}_{(2)}\cdot\mathbf{e}_a) = 0\\ &\mathbf{n}\cdot\mathbf{n}_{(1)} = 0 && \mathbf{n}_{(1)}\cdot\mathbf{n}_{(1)} + 2\mathbf{n}\cdot\mathbf{n}_{(2)} = 0\end{aligned}$$

which provide six equations for the six unknowns $\mathbf{n}_{(1)}, \mathbf{n}_{(2)}$. Some simple algebra gives

$$\widetilde{\mathbf{n}} = \mathbf{n} - \epsilon(\mathbf{n}\cdot\mathbf{W}_a)\mathbf{e}^a + \epsilon^2[(\mathbf{n}\cdot\mathbf{W}_b)(\mathbf{e}^b\cdot\mathbf{W}^a)\mathbf{e}_a - \tfrac{1}{2}(\mathbf{n}\cdot\mathbf{W}^a)(\mathbf{n}\cdot\mathbf{W}_a)\mathbf{n}] + \mathcal{O}(\epsilon^3). \tag{40}$$

Note that if the deformation is such that $(\mathbf{n}\cdot\mathbf{W}_a) = 0$ the normals to the two surfaces coincide. This happens for parallel surfaces, defined by $\mathbf{W} = a\mathbf{n}$, with a constant [21].

The extrinsic curvature of $\widetilde{\Sigma}$ is $\widetilde{K}_{ab} := -\widetilde{\mathbf{n}}\cdot\partial_a\widetilde{\mathbf{e}}_b$. Expanding the right-hand side to second order in ϵ, we obtain

$$\widetilde{K}_{ab} = K_{ab} - \mathbf{n}_{(1)}\cdot\partial_a\mathbf{e}_b - \mathbf{n}\cdot\partial_a\mathbf{e}_{b(1)} - \mathbf{n}_{(1)}\cdot\partial_a\mathbf{e}_{(1)} - \mathbf{n}_{(2)}\cdot\partial_a\mathbf{e}_{b(1)} + \mathcal{O}(\epsilon^3). \tag{41}$$

We use (21), (40), together with the Gauss–Weingarten equation for Σ, (2). We obtain

$$\widetilde{K}_{ab} = K_{ab} - \epsilon(\mathbf{n}\cdot\nabla_a\mathbf{W}_b) + \epsilon^2\big[(\mathbf{n}\cdot\mathbf{W}_c)(\mathbf{e}^c\cdot\nabla_a\mathbf{W}_b) - \tfrac{1}{2}K_{ab}(\mathbf{n}\cdot\mathbf{W}_c)(\mathbf{n}\cdot\mathbf{W}^c)\big] + \mathcal{O}(\epsilon^3). \tag{42}$$

We note that $\widetilde{K}_{ab}$ transforms covariantly under reparametrizations of Σ and that it involves two derivatives of the deformation vector $\mathbf{W}$. Note that for parallel surfaces, one has that $K_{ab(1)} = \epsilon a K_a{}^c K_{cb}$ and $K_{ab(2)} = 0$.

For the trace of the extrinsic curvature K, equations (26), (42) imply

$$\begin{aligned}\widetilde{K} = K - \epsilon[(\mathbf{n}\cdot\nabla_a\mathbf{W}^a) + 2K^{ab}(\mathbf{e}_a\cdot\mathbf{W}_b)] + \epsilon^2[&2(\mathbf{e}^a\cdot\mathbf{W}^b)(\mathbf{n}\cdot\nabla_a\mathbf{W}_b)\\ &+ 2K_{ab}(\mathbf{e}^a\cdot\mathbf{W}_c)(\mathbf{e}^c\cdot\mathbf{W}^b) + K_{ab}(\mathbf{e}^a\cdot\mathbf{W}_c)(\mathbf{e}^b\cdot\mathbf{W}^c) + (\mathbf{n}\cdot\mathbf{W}_c)(\mathbf{e}^c\cdot\nabla_a\mathbf{W}^a)\\ &- K_{ab}(\mathbf{n}\cdot\mathbf{W}^a)(\mathbf{n}\cdot\mathbf{W}^b) - \frac{K}{2}(\mathbf{n}\cdot\mathbf{W}^c)(\mathbf{n}\cdot\mathbf{W}_c)] + \mathcal{O}(\epsilon^3).\end{aligned} \tag{43}$$

At this point we can use the Gauss–Codazzi equation (12) to check the validity of these expressions for the deformation of the extrinsic curvature. At first order, using (26), (42), (43), we reproduce (37). Note that the check is non-trivial; it requires extensive use, in various degrees of contraction, of the Gauss–Codazzi–Mainardi equations.

5. Deformations of the geometry: decomposed

The existence of a basis adapted to the surface Σ, $\{\mathbf{e}_a, \mathbf{n}\}$, suggests a natural decomposition of the deformation vector $\mathbf{W}$ into its tangential and normal components (see e.g. [11]),

$$\mathbf{W} = \Phi^a \mathbf{e}_a + \Phi \mathbf{n}. \tag{44}$$

It follows that the projections of the first two derivatives of the deformation vector are

$$\mathbf{e}_a \cdot \mathbf{W}_b = \nabla_b \Phi_a + \Phi K_{ab} \tag{45}$$

$$\mathbf{n} \cdot \mathbf{W}_a = \nabla_a \Phi - \Phi^b K_{ab} \tag{46}$$

$$\mathbf{e}_c \cdot \nabla_a \mathbf{W}_b = \nabla_a \nabla_b \Phi_c - \Phi^d K_{bd} K_{ac} + 2K_{c(a} \nabla_{b)} \Phi + \Phi \nabla_a K_{bc} \tag{47}$$

$$\mathbf{n} \cdot \nabla_a \mathbf{W}_b = \nabla_a \nabla_b \Phi - \Phi K_{ac} K^c{}_b - 2K_{c(a} \nabla_{b)} \Phi^c - \Phi^c \nabla_c K_{ab} \tag{48}$$

where we have used the Gauss–Weingarten equations (2), (3), and the contracted Codazzi–Mainardi equation (13).

Using these expressions, the basic geometric quantities that characterize $\widetilde{\Sigma}$ take the form, to first order in ϵ,

$$g_{ab(1)} = \epsilon(2K_{ab}\Phi + 2\nabla_{(a}\Phi_{b)}) \tag{49}$$

$$\sqrt{g}_{(1)} = \epsilon\sqrt{g}(K\Phi + \nabla_a \Phi^a) \tag{50}$$

$$K_{ab(1)} = \epsilon\left(-\nabla_a \nabla_b \Phi + K_{ac} K^c{}_b \Phi + \Phi^c \nabla_c K_{ab} + 2K_{c(a} \nabla_{b)} \Phi^c\right) \tag{51}$$

$$K_{(1)} = \epsilon(-\nabla^2 \Phi - K_{ab} K^{ab} \Phi + \Phi^c \nabla_c K) \tag{52}$$

$$\mathcal{R}_{(1)} = \epsilon[2(K^{ab} - K g^{ab}) \nabla_a \nabla_b \Phi + \Phi^c \nabla_c \mathcal{R}]. \tag{53}$$

We can thus identify both a normal (linear in Φ), and a tangential deformation (linear in Φ^a) of these geometrical quantities. We note that these expressions coincide with the ones obtained, e.g., in [11].

In particular, to first order the tangential components correspond to an infinitesimal (active) reparametrization of the surface. Indeed, each of the three surface scalars $\mathbf{X}(\xi^a)$ transforms as

$$\delta_{\mathsf{rep}} \mathbf{X} = v^a(\xi^b) \partial_a \mathbf{X} = v^a \mathbf{e}_a \tag{54}$$

under a reparametrization $\xi^a \to \xi^a - v^a$. This is exactly the effect of a tangential deformation at first order with the identification of v^a with the surface vector field defined by the projection $\Phi^a = \mathbf{W} \cdot \mathbf{e}^a$. Reflecting this fact, at first order, geometrical quantities transform as a surface Lie derivative along the surface vector field Φ^a. For example, setting $\Phi = 0$ in (49), we have that

$$g_{ab(1)\mathsf{tang.}} = 2\nabla_{(a}\Phi_{b)} = \mathcal{L}_{\Phi^a} g_{ab} \tag{55}$$

where $\mathcal{L}_{\Phi^a}$ denotes the surface Lie derivative along Φ^a.

It is important to emphasize that, having made this identification at first order, the tangential part of any total geometrical invariant of $\widetilde{\Sigma}$ is always given by the integral of a total divergence, which vanishes over a closed surface without boundaries. To see this,

consider an invariant $I = \int \mathrm{d}A\, f(\xi^a)$, where $f(\xi^a)$ is a scalar under reparametrizations. Since f is a scalar we have simply that

$$f_{(1)\text{tang.}} = \Phi^a \partial_a f \tag{56}$$

no matter how complicated the dependence of f on the geometry might be. Moreover, setting $\Phi = 0$ in (50), we have $\sqrt{g}_{(1)\text{tang.}} = \sqrt{g}\nabla_a \Phi^a$, and therefore

$$I_{(1)\text{tang.}} = \int \mathrm{d}A \nabla_a (\Phi^a f). \tag{57}$$

At first order, we can always disentangle the physical normal deformation from reparametrizations and we can safely set Φ^a to vanish. Matters, however, are not so simple at second order. Let us consider the second-order variation of the metric. Using the completeness relation (1) it takes the form

$$\begin{aligned} g_{ab(2)} &= \epsilon^2[(\mathbf{e}_a \cdot \mathbf{W}_b)(\mathbf{e}^a \cdot \mathbf{W}^b) + (\mathbf{n} \cdot \mathbf{W}_a)(\mathbf{n} \cdot \mathbf{W}^a)] \\ &= \epsilon^2[(\nabla_b \Phi_a + \Phi K_{ab})(\nabla^b \Phi^a + \Phi K^{ab}) + (\nabla_a \Phi - \Phi^c K_{ac})(\nabla^a \Phi - \Phi_d K^{ad})] \\ &= \epsilon^2\big[\nabla_a \Phi \nabla_b \Phi + K_a{}^c K_{cb} \Phi^2 + 2K_{c(a}\nabla_{b)}(\Phi \Phi^c) + \nabla_a \Phi^c \nabla_b \Phi_c + K_{ac} K_{bd} \Phi^c \Phi^d\big] \end{aligned} \tag{58}$$

and with (49) this completely describes the deformation of the metric to all orders. At second order, normal and tangential deformations begin to talk to each other. When both normal and tangential deformations are present, there is a mixing. The purely tangential deformation at this order is certainly not simply a second-order reparametrization, in the sense of a composition of Lie derivatives. This might appear to be obvious: the second-order tangential deformation involves the extrinsic geometry through the quadratic term in K_{ab}, whereas a reparametrization is a purely intrinsic concept and as such it should not involve K_{ab}. One has to be careful, however, to check how the dependence on the extrinsic geometry enters.

Let us look more closely at the issue of reparametrization covariance at second order. Consider an infinitesimal change of coordinates on the surface $\xi^a \to \xi'^a = \xi^a + v^a$, and a surface scalar field $f(\xi^a)$. By the definition of scalar field we have $f'(\xi'^a) = f(\xi^a)$. In order to evaluate the change in the scalar field at the same point, we expand $f'(\xi'^a) = f'(\xi^a + v^a)$ in powers of v^a,

$$f'(\xi'^a) = f'(\xi^a) + v^a \partial_a f'(\xi^a) + \tfrac{1}{2}(v^a \partial_a)(v^b \partial_b) f(\xi^a) + \cdots \tag{59}$$

where we replace f' by f in the last term since it is already of second order in v^a. For the middle term note that

$$\begin{aligned} v^a \partial_a f'(\xi^a) &= v^a \partial_a [f'(\xi'^a - v^a)] \\ &= v^a \partial_a [f'(\xi'^a) - v^b \partial_b f(\xi^a)] \\ &= v^a \partial_a [f(\xi^a) - v^b \partial_b f(\xi^a)]. \end{aligned} \tag{60}$$

Therefore, to second order, we have

$$\delta_{\text{rep}} f(\xi^a) = f'(\xi^a) - f(\xi^a) = -v^a \partial_a f(\xi^a) + \tfrac{1}{2}(v^a \partial_a)(v^b \partial_b) f(\xi^a). \tag{61}$$

At first order, we have minus the Lie derivative of the scalar field, since now we are considering passive transformations. The second-order contribution is the composition of two Lie derivatives.

In particular, for the embedding functions we obtain

$$\delta_{\text{rep}} \mathbf{X} = -v^a \mathbf{e}_a + \tfrac{1}{2}(v^a \nabla_a v^b \mathbf{e}_b - v^a v^b K_{ab} \mathbf{n}). \tag{62}$$

The important point here is that, at second order, a reparametrization will generally alter the embedding functions. In contrast, by construction $\mathbf{X}$ is only modified at first order in $\mathbf{W}$.

Note also that, at second order, a reparametrization produces a change in $\mathbf{X}$ along the *normal* direction. Moreover, it depends explicitly on the extrinsic geometry. This justifies our earlier caveat.

The tangent vectors $\mathbf{e}_a$ transform as covariant surface vectors under reparametrization: we have at first order for a vector f_a

$$\delta_{\rm rep} f_{a(1)} = -v^b \partial_b f_a - \partial_a v^b f_b. \tag{63}$$

Thus for $\mathbf{e}_a$:

$$\begin{aligned}\delta_{\rm rep}\mathbf{e}_{a(1)} &= -v^b\partial_b\mathbf{e}_a - \partial_a v^b \mathbf{e}_b \\ &= -v^b\nabla_b\mathbf{e}_a - \nabla_a v^b\mathbf{e}_b \\ &= v^b K_{ab}\mathbf{n} - \nabla_a v^b \mathbf{e}_b\end{aligned} \tag{64}$$

involving both tangential and normal parts. Note how the normal contribution projects out of $\delta_{\rm rep} g_{ab(1)} = 2\mathbf{e}_{(a}\cdot\delta_{\rm rep}\mathbf{e}_{b)(1)} = 2\nabla_{(a}v_{b)}$. At second order,

$$\delta_{\rm rep}\mathbf{e}_{a(2)} = \tfrac{1}{2}(-v^b\partial_b\delta_{\rm rep}\mathbf{e}_{a(1)} - \delta_{\rm rep}\mathbf{e}_{b(1)}\partial_a v^b). \tag{65}$$

Similarly, we obtain for the metric at second order

$$\delta_{\rm rep} g_{ab(2)} = \tfrac{1}{2}(v^c(\nabla_c\nabla_a v_b + \nabla_c\nabla_b v_a) + (\nabla_c v_a)(\nabla_b v^c) + (\nabla_c v_b)(\nabla_a v^c) + 2(\nabla_a v_c)(\nabla_b v^c). \tag{66}$$

Note that $\delta_{\rm rep} g_{ab(2)}$ is manifestly intrinsic and, as such, distinct from the contribution to $g_{ab(2)}$ quadratic in Φ^a on setting $\Phi^a = v^a$. We note also that this expression coincides with the metric at second order induced by (62)

$$\delta_{\rm rep} g_{ab(2)} = 2\mathbf{e}_{(a}\cdot\delta_{\rm rep}\mathbf{e}_{b)(2)} + \delta_{\rm rep}\mathbf{e}_{a(1)}\cdot\delta_{\rm rep}\mathbf{e}_{b(1)}. \tag{67}$$

At second order, we see that we cannot disentangle the physical normal deformation from reparametrizations and we cannot set Φ^a to vanish. However, as we will see below in section 8, when considering the second-order deformation of global geometric invariants, the tangential component of the deformation will appear only in boundary terms or in terms that vanish when the membrane is at equilibrium.

6. Variation of the Helfrich–Canham Hamiltonian: first order

Let us expand the Hamiltonian $F[\mathbf{X}]$ as given by (19) to first order in ϵ. This will allow us to identify its Euler–Lagrange derivative, and the equilibrium conditions for the vesicle. (For a different approach emphasizing the normal component of the deformation, see [11].)

The expansion can always be written in the form

$$F_{(1)}[\mathbf{X},\mathbf{W}] = \epsilon\int {\rm d}A\,\mathbf{E}_F\cdot\mathbf{W} + \epsilon\int {\rm d}A\nabla_a\mathcal{Q}^a. \tag{68}$$

Here $\mathbf{E}_F$ denotes the Euler–Lagrange derivative for $F[\mathbf{X}]$. The quantity $\mathcal{Q}^a$ appearing in the total divergence in the second term is the Noether current [16], which will be used in section 7 to derive the stresses and torques acting on the surface associated with $F[\mathbf{X}]$.

We use the results of section 2 to derive, term by term, the various contributions to (68). For the area of the vesicle, we find using (23) that

$$A_{(1)} = \epsilon\int {\rm d}A(\mathbf{e}_a\cdot\mathbf{W}^a). \tag{69}$$

To cast this expression in the form (68), we integrate by parts, and obtain

$$A_{(1)} = \epsilon\int {\rm d}A\,K\mathbf{n}\cdot\mathbf{W} + \epsilon\int {\rm d}A\,\nabla_a(\mathbf{e}^a\cdot\mathbf{W}). \tag{70}$$

Therefore, the Euler–Lagrange derivative of the area is purely normal, and proportional to the mean extrinsic curvature, $\mathbf{E}_A = E_A\mathbf{n} = K\mathbf{n}$. The first feature is common to all reparametrization invariants: to first order in ϵ, tangential deformations contribute only boundary terms, as shown in the previous section. The latter tells us that minimal surfaces, extremizing the area, have vanishing mean extrinsic curvature, $E_A = K = 0$.

Note that for a constant normal displacement $\mathbf{W} = a\mathbf{n}$ we have

$$A_{(1)} = a\int \mathrm{d}A\, K = aM \tag{71}$$

so that the total mean extrinsic curvature is proportional to the area difference in the normal direction.

If we require that the area be infinitesimally locally invariant, then we have a constraint on $\mathbf{W}$ of the form $\mathbf{e}_a \cdot \mathbf{W}^a = 0$. This does not, however, alter the value of the Euler–Lagrange derivative.

For the volume enclosed by the vesicle, we use definition (17), together with (23), (40), to derive

$$\begin{aligned} V_{(1)} &= \frac{\epsilon}{3}\left[\int \mathrm{d}A_{(1)}(\mathbf{n}\cdot\mathbf{X}) + \int \mathrm{d}A(\mathbf{n}_{(1)}\cdot\mathbf{X} + \epsilon\mathbf{n}\cdot\mathbf{W})\right] \\ &= \frac{\epsilon}{3}\int \mathrm{d}A[(\mathbf{W}\cdot\mathbf{n}) + (\mathbf{e}_a\cdot\mathbf{W}^a)(\mathbf{n}\cdot\mathbf{X}) - (\mathbf{n}\cdot\mathbf{W}^a)(\mathbf{e}_a\cdot\mathbf{X})]. \end{aligned}$$

We integrate by parts the second and third terms and neglect a total divergence to obtain

$$V_{(1)} = \epsilon\int \mathrm{d}A\,\mathbf{n}\cdot\mathbf{W} \tag{72}$$

therefore we find that the Euler–Lagrange derivative of the enclosed volume functional is simply unity, $E_V = 1$.

Let us consider now the total mean extrinsic curvature, M, as defined in (18). We use (23) and (43) to derive

$$M_{(1)} = \epsilon\int \mathrm{d}A[K(\mathbf{e}^a\cdot\mathbf{W}_a) - 2K^{ab}(\mathbf{e}_a\cdot\mathbf{W}_b) - (\mathbf{n}\cdot\nabla_a\mathbf{W}^a)].$$

To put it in the form (68), we integrate by parts and use the Gauss–Weingarten equations (2), (3) to arrive at

$$M_{(1)} = \epsilon\int \mathrm{d}A\,\mathcal{R}(\mathbf{n}\cdot\mathbf{W}) + \epsilon\int \mathrm{d}A\,\nabla_a[(Kg^{ab} - K^{ab})(\mathbf{e}_b\cdot\mathbf{W}) - (\mathbf{n}\cdot\mathbf{W}^a)]. \tag{73}$$

The scalar intrinsic curvature appears as the Euler–Lagrange derivative of the total mean extrinsic curvature functional $E_M = \mathcal{R}$.

As mentioned above, the Gaussian bending energy F_G as given by (15) is a topological invariant. As such, we expect both $F_{G(1)}$ and $F_{G(2)}$ to be total divergences. As this provides a non-trivial check, let us consider its expansion to first order. Moreover, in any case, we are interested in the non-vanishing contribution to the Noether charge. Using (98), (37), we have

$$F_{G(1)} = \epsilon\alpha_G\int \mathrm{d}A[2(K^{ab} - Kg^{ab})(\mathbf{n}\cdot\nabla_a\mathbf{W}_b) - \mathcal{R}(\mathbf{e}^a\cdot\mathbf{W}_a)]. \tag{74}$$

We integrate the first two terms by parts, and use the the second Gauss–Weingarten equation (3), together with the contracted Codazzi–Mainardi equations (13), to obtain

$$\begin{aligned} F_{G(1)} = \epsilon\alpha_G &\int \mathrm{d}A\big[-\mathcal{R}g^{ab} + 2KK^{ab} - 2K^{ac}K^b{}_c\big](\mathbf{e}_a\cdot\mathbf{W}_b) \\ &+ \epsilon\alpha_G\int \mathrm{d}A\,\nabla_a[2(K^{ab} - Kg^{ab})(\mathbf{n}\cdot\mathbf{W}_b) - \mathcal{R}(\mathbf{e}^a\cdot\mathbf{W})] \end{aligned} \tag{75}$$

where the first line vanishes because of the contracted Gauss–Codazzi equation (11). Therefore $F_{G(1)}$ is given by a total divergence. With the help of the relationship (38), the second-order term $F_{G(2)}$ is a total divergence as well. To see this, recall the fact that the 'deformation' of the divergence of a vector density is equal to the divergence of the deformation. In this particular case we have

$$F_{G(1)} = \epsilon\alpha_G \int \mathrm{d}A\, \nabla_a \mathcal{Q}^a_G = \epsilon\alpha_G \int \mathrm{d}^2\xi\, \partial_a\big(\sqrt{g}\,\mathcal{Q}^a_G\big) \tag{76}$$

where $\mathcal{Q}^a_G$ denotes the contribution of the Gaussian bending rigidity to the Noether current, and it is given explicitly by the argument of the covariant derivative in (75). At second order we have then the total divergence

$$F_{G(2)} = \frac{1}{2}\widetilde{F}_{G(1)} = \frac{\epsilon}{2}\alpha_G \int \mathrm{d}^2\xi\, \partial_a\big[\big(\sqrt{g}\mathcal{Q}^a_G\big)_{(1)}\big] \tag{77}$$

Finally, for the bending energy (14), we obtain, using (23), (43),

$$F_{b(1)} = \epsilon \int \mathrm{d}A[K^2(\mathbf{e}_a\cdot\mathbf{W}^a) - 4KK^{ab}(\mathbf{e}_a\cdot\mathbf{W}_b) - 2K(\mathbf{n}\cdot\nabla_a\mathbf{W}^a)] \tag{78}$$

and integration by parts twice gives

$$\begin{aligned} F_{b(1)} = \epsilon \int \mathrm{d}A[-2\nabla^2 K + K^3 - 2KK_{ab}K^{ab}]\mathbf{n}\cdot\mathbf{W} \\ + \epsilon \int \mathrm{d}A\, \nabla_a[(K^2 g^{ab} - 2KK^{ab})(\mathbf{e}_b\cdot\mathbf{W}) + 2(\nabla^a K)(\mathbf{n}\cdot\mathbf{W}) - 2K(\mathbf{n}\cdot\mathbf{W}^a)]. \end{aligned} \tag{79}$$

Therefore the Euler–Lagrange derivative of the bending energy functional is

$$\begin{aligned} E_{F_b} &= -2\nabla^2 K + K^3 - 2KK_{ab}K^{ab} \\ &= -2\nabla^2 K + K(2\mathcal{R} - K^2) \end{aligned} \tag{80}$$

where we have used the Gauss–Codazzi equation (12) to obtain the second line.

We are now in a position to write down the equilibrium conditions for the Hamiltonian (19). We set

$$\mathbf{E}_F = E_F\mathbf{n} \tag{81}$$

where

$$E_F = \alpha\left[-2\nabla^2 K + K(2\mathcal{R} - K^2)\right] + \mu K + \beta\mathcal{R} - P. \tag{82}$$

Then equilibrium configurations that extremize the Helfrich–Canham Hamiltonian satisfy

$$E_F = 0. \tag{83}$$

This is known as the shape equation [14]. Note that it is a nonlinear fourth-order partial differential equation. Progress in the understanding of its space of solutions has been limited to the special case of axisymmetric configurations, see e.g. [20].

7. Stresses and torques

The Noether current appearing in (68) allows the derivation of the stresses and the torques acting on the membrane, using the invariance under rigid motions in space. This was done in [16] by decomposing the deformation into its normal and tangential parts. Here, we provide an alternative derivation, which has the advantage of being more direct. See also [22, 23].

Before, we proceed, for the purposes of this section it is convenient to rewrite the Helfrich–Canham Hamiltonian (19) by isolating the volume term as

$$F = F_s - PV \tag{84}$$

with the surface part of the Hamiltonian $F_s = F_b + F_G + \mu A + \beta M$.

Collecting the various total surface divergences appearing in (70), (73), (75), (79), it is straightforward to identify the Noether current associated with F_s,

$$\begin{aligned}\mathcal{Q}^a = \alpha[&(K^2 g^{ab} - 2KK^{ab})(\mathbf{e}_b \cdot \mathbf{W}) + 2(\nabla^a K)(\mathbf{n} \cdot \mathbf{W}) - 2K(\mathbf{n} \cdot \mathbf{W}^a)] \\ &+ \beta[(Kg^{ab} - K^{ab})(\mathbf{e}_b \cdot \mathbf{W}) - \mathbf{n} \cdot \mathbf{W}^a] + \mu(\mathbf{e}^a \cdot \mathbf{W}) \\ &+ 2\alpha_G(K^{ab} - Kg^{ab})(\mathbf{n} \cdot \mathbf{W}_b).\end{aligned} \tag{85}$$

Note that although the Gaussian bending energy does not contribute to the Euler–Lagrange equations, it does appear in the Noether current $\mathcal{Q}^a$.

We consider now a (simply connected) piece of the membrane, which we denote by Σ_0, bounded by a curve C, and we specialize the variation of the Hamiltonian (68) to this arbitrary region of the membrane. We have that

$$F_{s(1)} = \epsilon \int_{\Sigma_0} \mathrm{d}A[E_{F_s}\mathbf{n} \cdot \mathbf{W} + \nabla_a \mathcal{Q}^a]. \tag{86}$$

We exploit the invariance of the Hamiltonian under rigid motions in space. First, we consider an infinitesimal translation $\mathbf{W} = \mathbf{a}$, with $\mathbf{a}$ constant. As the Hamiltonian is invariant under translations, the left-hand side of (86) vanishes, and with the stress tensor $\mathbf{f}^a$ defined by

$$\mathcal{Q}^a = -\mathbf{a} \cdot \mathbf{f}^a \tag{87}$$

it follows that we can write the Euler–Lagrange derivative as a conservation law

$$E_{F_s}\mathbf{n} = \nabla_a \mathbf{f}^a \tag{88}$$

where the stresses associated with the Hamiltonian (19) are given by

$$\mathbf{f}^a = -\alpha[(K^2 g^{ab} - 2KK^{ab})\mathbf{e}_b + 2(\nabla^a K)\mathbf{n}] - \beta(Kg^{ab} - K^{ab})\mathbf{e}_b - \mu\mathbf{e}^a. \tag{89}$$

We emphasize that it is far from obvious from the shape equation itself (83) that it can be written as a conservation law.

There are three conservation laws, and only one shape equation. This is a consequence of the reparametrization invariance of the Hamiltonian. This statement can be made explicit using the decomposition of the stress tensor into tangential and normal parts as follows:

$$\mathbf{f}^a = f^{ab}\mathbf{e}_b + f^a\mathbf{n}. \tag{90}$$

The surface covariant derivative then gives

$$\nabla_a f^a - K_{ab} f^{ab} = E_{F_s} = P \tag{91}$$

$$\nabla_a f^{ab} + K^b{}_a f^a = 0. \tag{92}$$

The first equation is the shape equation expressed in terms of the projections f^a and f^{ab}. The second equation expresses the content of reparametrization invariance as a consistency check: the normal stress and the tangential stress must balance exactly in this way. Note that this identity is potentially useful in numerical simulations, where reparametrization invariance is necessarily lost, and one is interested in quantifying the degree of violation.

The physical meaning of the stress tensor $\mathbf{f}^a$ is perhaps best illustrated by considering the total force per unit length acting on the curve C. Concretely, C may be the shape of an edge of the membrane [24], or the line boundary between the two phases of a two-component

vesicle [25]. If we consider a basis $\{\mathbf{t}, \mathbf{l}\}$ on the surface adapted to the curve C that bounds Σ_0, with $\mathbf{t}$ tangent to C, and $\mathbf{l} = l^a \mathbf{e}_a$ the (outward) normal to C on the surface, we obtain the force per unit length acting on C, $l_a \mathbf{f}^a = \mathbf{f}$, as

$$\mathbf{f} = [K_{\parallel\perp}(2\alpha K + \beta)]\mathbf{t} + [\alpha K(K_\perp - K_\parallel) - \beta K_\parallel - \mu]\mathbf{l} - 2\alpha(\nabla_\perp K)\mathbf{n} \tag{93}$$

where we denote the projections of the extrinsic curvature onto the surface as $K_\parallel = K_{ab}t^a t^b$, $K_\perp = K_{ab}l^a l^b$ and $K_{\perp\parallel} = K_{ab}t^a l^b$. Note that $K = K_\perp + K_\parallel$, and $\mathcal{R} = 2\big(K_\parallel K_\perp - K_{\perp\parallel}^2\big)$. $\nabla_\perp = l^a \nabla_\perp$ denotes the covariant derivative along the direction normal to the curve C.

Similarly as we showed for translations, for an infinitesimal rotation of the form $\mathbf{W} = \mathbf{b} \times \mathbf{X}$, we can obtain the torques acting on the surface associated with the Hamiltonian (19). We define the total angular momentum $\mathbf{m}^a$

$$\mathcal{Q}^a = -\mathbf{b} \cdot \mathbf{m}^a \tag{94}$$

where the torque $\mathbf{m}^a$ can be split into its 'orbital' and 'differential' parts as

$$\mathbf{m}^a = \mathbf{X} \times \mathbf{f}^a + \mathbf{s}^a. \tag{95}$$

From the Noether charge (85) we obtain directly that the terms contributing to $\mathbf{s}^a$ are the ones involving derivatives of the deformation vector, $\mathbf{W}_a$, so that

$$\mathbf{s}^a = [(2\alpha K + \beta)g^{ab} + 2\alpha_G(K g^{ab} - K^{ab})]\mathbf{e}_b \times \mathbf{n}. \tag{96}$$

Note that it is tangential to the surface.

The differential torque and the stress tensor are related by [16]

$$\nabla_a \mathbf{s}^a = \mathbf{f}^a \times \mathbf{e}_a. \tag{97}$$

We emphasize that this expression is also valid when not in equilibrium.

8. Variation of the Helfrich–Canham Hamiltonian: second order

In this section, we exploit the general results of sections 2 and 3 to derive the expansion to second order in ϵ of the Hamiltonian (19). As we did at first order, we derive the various terms that contribute to it. There are two possible strategies: on the one hand we can perform a direct expansion, alternatively we can exploit the identity (38), and deform the first-order terms we have obtained in section 6. We will adopt the most convenient strategy for each term. This part is a straightforward calculation. However, both in order to have a better understanding of the final result and to make contact with the variational approach of [11], we decompose the deformation vector $\mathbf{W}$ into components. This is a straightforward calculation as well, but it turns out that it is possible to organize the result in a way which isolates boundary terms and a contribution proportional to the Euler–Lagrange derivative of each term.

For the area, we have immediately, using (23), that

$$A_{(2)} = \frac{\epsilon^2}{2} \int \mathrm{d}A[(\mathbf{n} \cdot \mathbf{W}^a)(\mathbf{n} \cdot \mathbf{W}_a) + (\mathbf{e}_a \cdot \mathbf{W}^a)(\mathbf{e}_b \cdot \mathbf{W}^b) - (\mathbf{e}_b \cdot \mathbf{W}^a)(\mathbf{e}_a \cdot \mathbf{W}^b)]. \tag{98}$$

The direct expansion at second order of the volume is quite complicated. It is preferable to expand the first-order term, so that, using the identity (38) and (72), we have

$$\begin{aligned} V_{(2)} &= \frac{\epsilon}{2}\tilde{V}_{(1)} = \frac{\epsilon}{2}\left[\int \mathrm{d}A_{(1)}(\mathbf{W} \cdot \mathbf{n}) + \int \mathrm{d}A(\mathbf{W} \cdot \mathbf{n}_{(1)})\right] \\ &= \frac{\epsilon^2}{2} \int \mathrm{d}A[(\mathbf{e}_a \cdot \mathbf{W}^a)(\mathbf{W} \cdot \mathbf{n}) - (\mathbf{e}_a \cdot \mathbf{W})(\mathbf{W}^a \cdot \mathbf{n})]. \end{aligned} \tag{99}$$

Let us consider the second-order term in the expansion of the total mean curvature M. We have that

$$M_{(2)} = \int [\mathrm{d}A_{(2)} K + \mathrm{d}A_{(1)} K_{(1)} + \mathrm{d}A\, K_{(2)}] \tag{100}$$

so that using (23) and (43), we obtain

$$\begin{aligned} M_{(2)} = \epsilon^2 \int \mathrm{d}A \Bigg[& \frac{1}{2}(\mathbf{e}_a \cdot \mathbf{W}^a)^2 K - \frac{1}{2}(\mathbf{e}_a \cdot \mathbf{W}_b) K (\mathbf{e}^b \cdot \mathbf{W}^a) - (\mathbf{e}_a \cdot \mathbf{W}^a)(\mathbf{n} \cdot \nabla_a \mathbf{W}^a) \\ & - 2K^{bc}(\mathbf{e}_a \cdot \mathbf{W}^a)(\mathbf{e}_b \cdot \mathbf{W}_c) + 2(\mathbf{e}^a \cdot \mathbf{W}^b)(\mathbf{n} \cdot \nabla_a \mathbf{W}_b) \\ & + 2K^{ab}(\mathbf{e}_a \cdot \mathbf{W}_c)(\mathbf{e}^c \cdot \mathbf{W}_b) + K^{ab}(\mathbf{e}_a \cdot \mathbf{W}_c)(\mathbf{e}_b \cdot \mathbf{W}^c) \\ & + (\mathbf{n} \cdot \mathbf{W}_c)(\mathbf{e}^c \cdot \nabla_a \mathbf{W}^a) - K_{ab}(\mathbf{n} \cdot \mathbf{W}^a)(\mathbf{n} \cdot \mathbf{W}^b) \Bigg]. \end{aligned} \tag{101}$$

There is no obvious simplification. On the other hand, using the identity (38) and (73), we obtain, up to a total divergence, the simpler expression

$$\begin{aligned} M_{(2)} = \frac{\epsilon}{2}\tilde{M}_{(1)} = \frac{\epsilon^2}{2} \int \mathrm{d}A \{ & 2(K^{ab} - K g^{ab})(\mathbf{n} \cdot \nabla_a \mathbf{W}_b)(\mathbf{n} \cdot \mathbf{W}) \\ & - \mathcal{R}[(\mathbf{n} \cdot \mathbf{W}_a)(\mathbf{e}^a \cdot \mathbf{W}) + (\mathbf{n} \cdot \mathbf{W})(\mathbf{e}^a \cdot \mathbf{W}_a)] \}. \end{aligned} \tag{102}$$

These two expressions differ only by a total divergence. However, it is quite involved to extract it from (101).

For the bending energy, we have that a direct expansion is the more convenient approach and with

$$F_{b(2)} = \int [\mathrm{d}A_{(2)} K^2 + 2\mathrm{d}A_{(1)} K K_{(1)} + \mathrm{d}A K_{(1)} K_{(1)} + 2\mathrm{d}A\, K K_{(2)}] \tag{103}$$

using (23), (43), we obtain

$$\begin{aligned} F_{b(2)} = \epsilon^2 \int \mathrm{d}A \Bigg[& (\mathbf{n} \cdot \nabla_a \mathbf{W}^a)^2 + 2(2K^{ab} - K g^{ab})(\mathbf{e}_a \cdot \mathbf{W}_b)(\mathbf{n} \cdot \nabla_c \mathbf{W}^c) \\ & + 4K(\mathbf{e}^a \cdot \mathbf{W}^b)(\mathbf{n} \cdot \nabla_a \mathbf{W}_b) + 2K(\mathbf{n} \cdot \mathbf{W}^c)(\mathbf{e}_c \cdot \nabla_a \mathbf{W}^a) \\ & - 2K K^{ab}(\mathbf{n} \cdot \mathbf{W}_a)(\mathbf{n} \cdot \mathbf{W}_b) - \tfrac{1}{2}K^2(\mathbf{n} \cdot \mathbf{W}_a)(\mathbf{n} \cdot \mathbf{W}^a) \\ & + \big(4K^{ab}K^{cd} + 4K K^{ad} g^{bc} - 4K K^{cd} g^{ab} + 2K K^{ac} g^{bd} + \tfrac{1}{2}K^2 g^{ab} g^{cd} \\ & - \tfrac{1}{2}K^2 g^{ad} g^{bc}\big)(\mathbf{e}_a \cdot \mathbf{W}_b)(\mathbf{e}_c \cdot \mathbf{W}_d) \Big]. \end{aligned} \tag{104}$$

These expressions provide directly the second variation of the Helfrich–Canham Hamiltonian in terms of the deformation vector $\mathbf{W}$ and its first and second derivatives. It is desirable, however, in order to make contact with the expressions derived in [11], to decompose $\mathbf{W}$ into tangential and normal. This involves plugging (45) to (48) into the covariant expressions we have derived. For example, for the area functional we obtain

$$\begin{aligned} A_{(2)} = \frac{\epsilon^2}{2} \int \mathrm{d}A [& \nabla_a \Phi \nabla^a \Phi + (K^2 - K_{ab}K^{ab})\Phi^2 - 2K^{ab}\nabla(\Phi_b \Phi) + K_{ac}K_b{}^c \Phi^a \Phi^b \\ & + 2K\Phi\nabla_a \Phi^a + (\nabla_a \Phi^a)^2 - (\nabla_a \Phi^b)\nabla_b \Phi^a - 2K^{ab}\Phi\nabla_a \Phi_b]. \end{aligned} \tag{105}$$

This is not the most useful form, however. Integrating by parts and isolating a total divergence, we can write it down in an alternative way as

$$\begin{aligned} A_{(2)} = \frac{\epsilon^2}{2} \int \mathrm{d}A \{ & -\Phi\nabla^2\Phi - K_{ab}K^{ab}\Phi^2 + K(K\Phi^2 + K_{ab}\Phi^a\Phi^b - 2\Phi^a\nabla_a\Phi) \\ & + \nabla_a[\Phi\nabla^a\Phi + 2\Phi\Phi_b(K g^{ab} - K^{ab}) - \Phi^b\nabla_b\Phi^a + \Phi^a\nabla_b\Phi^b] \}. \end{aligned} \tag{106}$$

The first line is the normal part of the second deformation. The second line is proportional to the trace of the extrinsic curvature K. It is essential to recognize that K is the Euler–Lagrangian derivative for the area functional, $E_A = K$. Therefore, at equilibrium, the second line will vanish. The third line is a total divergence and, for a closed vesicle without boundaries, it can be set to vanish.

This example indicates that it is possible to obtain in a systematic way simpler expressions by isolating terms that are total divergences. Let us use expression (68) for the first variation of the Helfrich–Canham Hamiltonian (see [11] for an equivalent argument in an alternative language). Using the identity (38), we have that

$$F_{(2)}[\mathbf{X},\mathbf{W}] = \frac{\epsilon}{2}\widetilde{F}_{(1)}[\mathbf{X},\mathbf{W}] = \frac{\epsilon}{2}\left\{\int \mathrm{d}A\; E_{F(1)}(\mathbf{n}\cdot\mathbf{W}) + \int \mathrm{d}A\; E_F(\mathbf{n}_{(1)}\cdot\mathbf{W}) + \int \mathrm{d}A_{(1)}\; E_F(\mathbf{n}\cdot\mathbf{W}) + \int \mathrm{d}^2\xi\; \partial_a[(\sqrt{g}\mathcal{Q}^a)_{(1)}]\right\} \tag{107}$$

where we have rewritten the second term of (68) so that $\sqrt{g}\mathcal{Q}^a$ is a scalar density of weight one; its divergence is then independent of the affine connection, so that variation and derivation commute. $E_{F(1)}$ denotes the first-order variation of the Euler–Lagrange derivative appearing in (83). We now use (23), (40) to obtain

$$F_{(2)}[\mathbf{X},\mathbf{W}] = \frac{\epsilon^2}{2}\int \mathrm{d}A\; E_F[(\mathbf{e}_a\cdot\mathbf{W}^a)(\mathbf{n}\cdot\mathbf{W}) - (\mathbf{e}_a\cdot\mathbf{W})(\mathbf{n}\cdot\mathbf{W}^a)] + \frac{\epsilon}{2}\int \mathrm{d}A\; E_{F(1)}(\mathbf{n}\cdot\mathbf{W}) + \frac{\epsilon}{2}\int \mathrm{d}^2\xi\; \partial_a[(\sqrt{g}\mathcal{Q}^a)_{(1)}]. \tag{108}$$

We now use the results of section 4 to express in components the deformations. Furthermore, we split the first-order correction of the Euler–Lagrange derivative $E_{F(1)}$ into its normal and tangential parts as $E_{F(1)} = E_{F(1)\mathsf{perp}} + E_{F(1)\mathsf{tang}}$, where, since E_F is a scalar, $E_{F(1)\mathsf{tang}} = \Phi^a\nabla_a E_F$. It follows that we can rewrite (108) as

$$F_{(2)}[\mathbf{X},\mathbf{W}] = \frac{\epsilon^2}{2}\int \mathrm{d}A\; E_F[K\Phi^2 + \Phi\nabla_a\Phi^a - \Phi^a\nabla_a\Phi + K_{ab}\Phi^a\Phi^b] + \Phi\Phi^a\nabla_a E_F + \frac{\epsilon}{2}\int \mathrm{d}A\; E_{F(1)\mathsf{perp}}(\mathbf{n}\cdot\mathbf{W}) + \frac{\epsilon}{2}\int \mathrm{d}^2\xi\,\partial_a[(\sqrt{g}\mathcal{Q}^a)_{(1)}] \tag{109}$$

or as

$$F_{(2)}[\mathbf{X},\mathbf{W}] = \frac{\epsilon}{2}\int \mathrm{d}A\; E_{F(1)\mathsf{perp}}(\mathbf{n}\cdot\mathbf{W}) + \frac{\epsilon^2}{2}\int \mathrm{d}A\; E_F[K\Phi^2 - 2\Phi^a\nabla_a\Phi + K_{ab}\Phi^a\Phi^b] + \frac{\epsilon^2}{2}\int \mathrm{d}A\,\nabla_a(E_F\Phi\Phi^a) + \frac{\epsilon}{2}\int \mathrm{d}^2\xi\;\partial_a[(\sqrt{g}\mathcal{Q}^a)_{(1)}]. \tag{110}$$

The second line is a total divergence, and it can be set to vanish safely. In the first line, the term proportional to the Euler–Lagrange derivative E_F is surprisingly simple. We recognize the same structure that appears in the second variation of the area functional in the form (106).

When the shape equation $E_F = 0$ is satisfied, up to a total divergence, we have that the second variation is simply

$$F_{(2)}[\mathbf{X},\mathbf{W}] = \frac{\epsilon}{2}\int \mathrm{d}A\; E_{F(1)\mathsf{perp}}(\mathbf{n}\cdot\mathbf{W}). \tag{111}$$

If we set the tangential part of the deformation to vanish, $\Phi^a = 0$, then the second variation takes the form

$$F_{(2)}[\mathbf{X},\mathbf{W}] = \frac{\epsilon}{2}\int \mathrm{d}A[E_{F(1)\mathsf{perp}}(\mathbf{n}\cdot\mathbf{W}) + \epsilon E_F K(\mathbf{n}\cdot\mathbf{W})^2]. \tag{112}$$

Let us consider now the expressions in components of the remaining terms in the second variation of the Helfrich–Canham Hamiltonian in the form (110). For the volume we have that

$$V_{(2)} = \frac{\epsilon^2}{2}\int \mathrm{d}A[K(K\Phi^2 + K_{ab}\Phi^a\Phi^b - 2\Phi^a\nabla_a\Phi) + \nabla_a(\Phi\Phi^a)]. \tag{113}$$

This case is quite special, since the contribution of the volume to the Euler–Lagrange derivative is $E_V = 1$, therefore, for the volume, $E_{V(1)} = 0$.

For the total mean extrinsic curvature specializing to either components (101) or (102), we obtain, up to a total divergence,

$$M_{(2)} = \epsilon^2\int \mathrm{d}A[(K^{ab} - Kg^{ab})\Phi\nabla_a\nabla_b\Phi - \frac{1}{2}\mathcal{R}K\Phi^2 + \frac{\epsilon^2}{2}\int \mathrm{d}A\,\mathcal{R}(K\Phi^2 + K_{ab}\Phi^a\Phi^b - 2\Phi^a\nabla_a\Phi). \tag{114}$$

Note that the second line is proportional to the Euler–Lagrange derivative for M, since $E_M = \mathcal{R}$.

For the bending energy, a direct specialization to components of (104) produces 74 terms, and it is impossible to tell the trees from the forest. However, the general considerations that lead to (110) imply that the dependence on the tangential component of the deformation Φ^a is determined. Therefore, we can keep only the normal part of the deformation Φ, so that, up to a total divergence, we obtain

$$F_{b(2)} = \epsilon^2\int \mathrm{d}A\left\{(\nabla^2\Phi)^2 + \frac{1}{2}(K^2 - 2\mathcal{R})\Phi\nabla^2\Phi + 2KK^{ab}\Phi\nabla_a\nabla_b\Phi + K(\nabla_a K)(\nabla^a\Phi)\Phi - 2K^{ab}(\nabla_a K)(\nabla_b\Phi)\Phi + \left(K^4 - \frac{5}{2}K^2\mathcal{R} + \mathcal{R}^2\right)\Phi^2\right\}. \tag{115}$$

This corresponds to the form (112) of the second variation.

9. Concluding remarks

We have presented in some detail a fully covariant approach to the deformations of the Helfrich–Canham Hamiltonian; where applicable, we have compared it to the approach adopted in [11] in which tangential deformations were discarded.

On balance, we feel that there is something to be learnt from both approaches; the reader who has considered both may better judge which form of perturbation theory is more appropriate to the issue being addressed. For example, whereas it is trivially obvious in the covariant approach that the metric tensor is subject to variations of second order and no higher, once the decomposition has been affected, this fact becomes heavily disguised and would appear to involve miraculous cancellations. On the other hand, the second-order variation of the Hamiltonian about an equilibrium configuration is considerably more transparent when expressed in terms of normal deformations.

The nature of tangential deformations of individual geometrical tensors has been clarified at second order, an issue clearly beyond the scope of the analysis in [11] to address. We have shown giving explicit examples that, at this order and higher, tangential deformations are not reparametrizations. We suspect that there is much still to be pinned down on the issue. Even in the well-explored field of general relativity, disentangling coordinate artefacts from physical perturbations at second order remains a vexed issue [26].

We have not examined explicitly any order in perturbation theory higher than the second. This is not going to be simple; it remains a highly non-trivial challenge. The systematic approach we have outlined will, it is hoped, provide a few reliable signposts.

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